South Australian
Certificate of Education

1

Mathematical Methods
2019

Question booklet 1

- Questions 1 to 9 (72 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 20 if you need more space
- Allow approximately 95 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 190 minutes
Total marks: 146

Attach your SACE registration number label here

Graphics calculator

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   Model ____________

2. Brand ____________

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Question 1 (8 marks)

(a) For the functions below, determine $\frac{dy}{dx}$. You do not need to simplify your answers.

(i) $y = 3\cos(2 + 7x)$.

(2 marks)

(ii) $y = \frac{5x + 2}{x\sqrt{x}}$.

(3 marks)

(b) Evaluate the following integral:

$$\int \frac{2x^2 + x - 1}{x} \mathrm{d}x.$$

(3 marks)

Question 2 (7 marks)

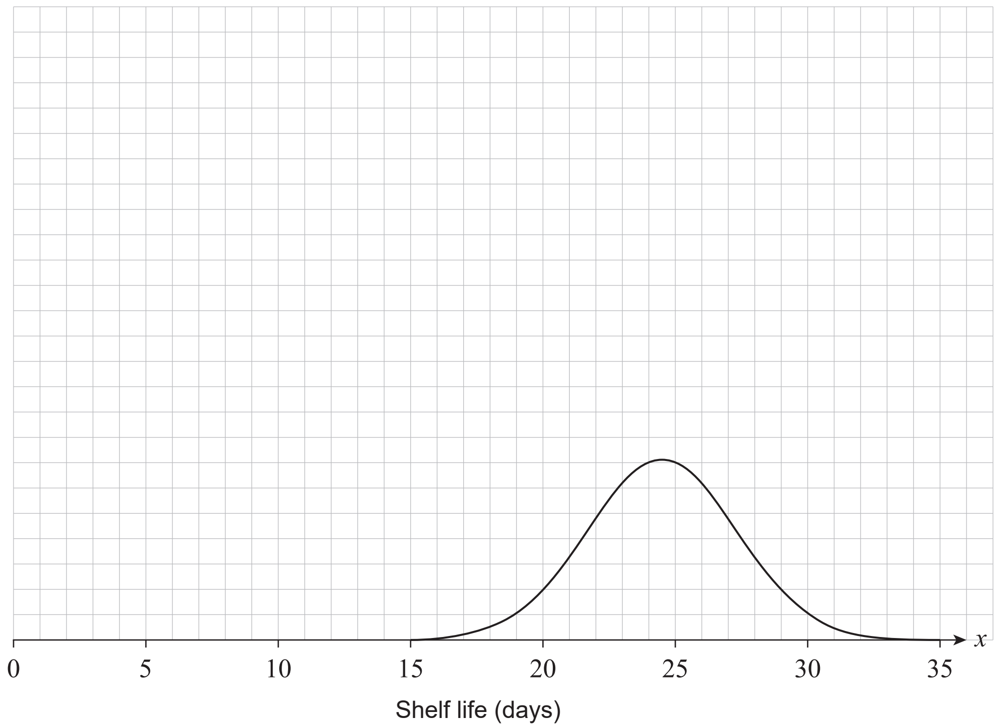
The 'shelf life' of a product is the recommended maximum time for which the product can be stored before becoming unfit for use, consumption, or sale.

When stored in the refrigerator, eggs have a shelf life that is normally distributed, with a mean of 24.5 days and a standard deviation of 2.8 days.

When stored in a cool place, bread has a shelf life that is normally distributed, with a mean of 2.4 days and a standard deviation of 0.8 days.

(a) Let E be the distribution of the shelf life of a randomly selected egg and B be the distribution of the shelf life of a randomly selected loaf of bread. The distribution of E is shown below.

On the axis provided below, sketch the distribution of B.



(2 marks)

(b) What is the probability that a randomly selected egg will have a shelf life of 3 weeks or less?

(1 mark)

(c) Ten per cent of eggs have a shelf life of k days or greater.

Find k.

(1 mark)

(d) A person is considering consuming a 36-day-old egg and some 5-day-old bread.

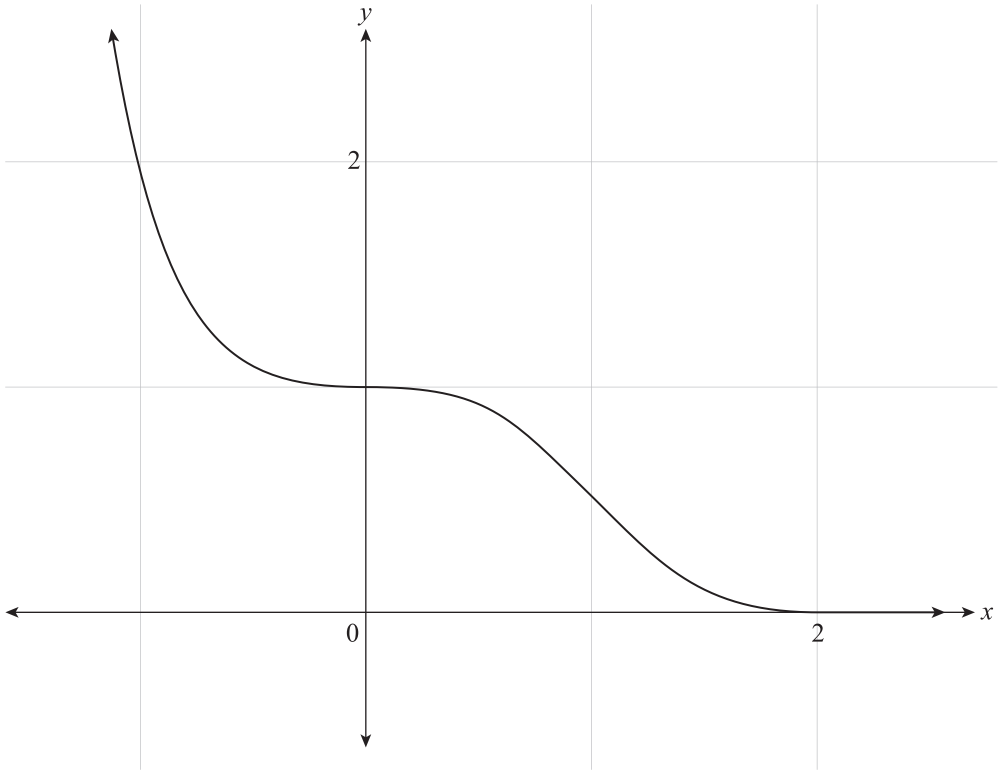
Assume that the egg and the bread have been stored appropriately and have been randomly selected.

Which *one* of these two products is more likely to be unfit for consumption? Support your answer with calculations.

(3 marks)

Question 3 (9 marks)

The graph of $f(x) = e^{-\frac{2}{3}x^3}$ is shown below.



(a) Find $f'(x)$.

(2 marks)

(b) Find $f''(x)$.

(2 marks)

(c) (i) Hence, using algebra, find the x-coordinates of all inflection points of the graph of $f(x) = e^{-\frac{2}{3}x^3}$.

(3 marks)

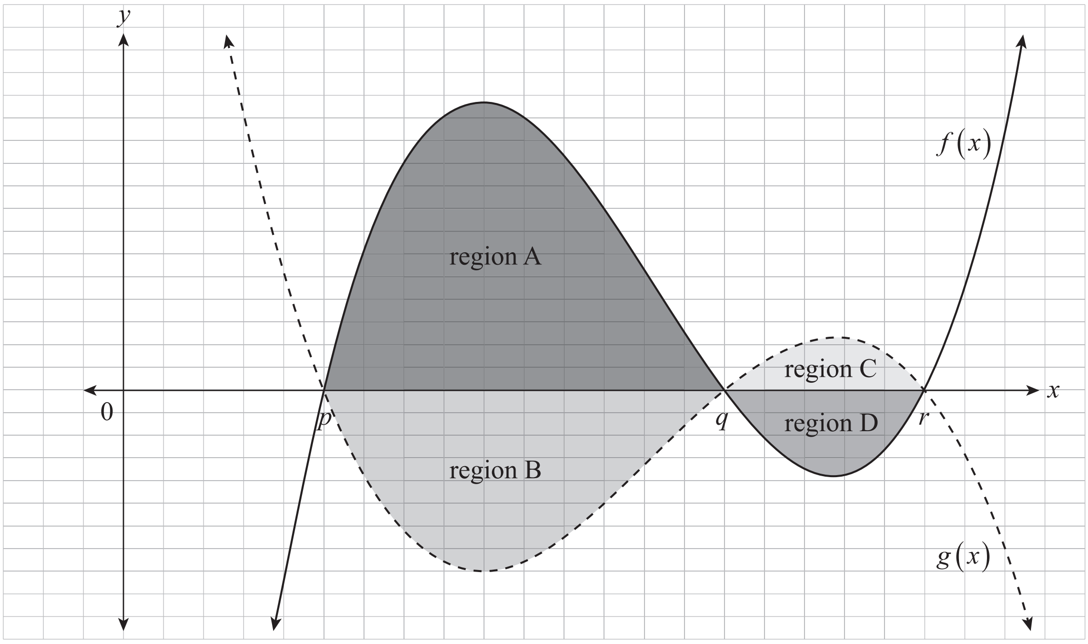
(ii) Using algebra, determine whether or not any of these inflection points are stationary inflection points.

(2 marks)

Question 4 (4 marks)

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.

The graphs intersect on the x-axis at points p, q, and r and form the regions that are labelled A, B, C, and D.



It is known that region A has an area of 16 square units, and that regions A and B have a combined area of 26 square units.

(a) State the value of $\int_p^q f(x)\,dx$.

(1 mark)

(b) State the value of $\int_p^q g(x)\,dx$.

(1 mark)

It is also known that $\int_p^r f(x)\,dx = 11$ and that $\int_p^r g(x)\,dx = -9$.

(c) Find the area of region C.

(1 mark)

(d) Calculate the value of $\int_p^r (f(x) - g(x))\,dx$.

(1 mark)

Question 5 (11 marks)

A group of conservationists is studying the number of nestlings (baby birds) in kookaburra nests throughout one region in South Australia. The conservationists find that, in this region, the number of nestlings, X, can be represented using the distribution below.

x	1	2	3	4	5	6
$\Pr(X = x)$	0.22	0.28	c	0.12	0.05	0.03

Source: adapted from Barcaldine Regional Council, www.abc.net.au

(a) Calculate c, the probability that a randomly selected kookaburra nest in this region contains three nestlings.

(1 mark)

(b) Find the mean number of nestlings per kookaburra nest in this region.

(1 mark)

The conservationists believe that during a period of drought, an entire family of kookaburras will survive *only* if their nest contains three or fewer nestlings.

(c) Calculate the probability that a randomly chosen kookaburra nest in this region contains three or fewer nestlings.

(1 mark)

(d) The conservationists want to understand the likelihood that entire families of kookaburras will survive a period of drought. They randomly select a sample of 20 kookaburra nests in this region and record the number of nestlings in each nest.

(i) Let Y represent the number of kookaburra nests in the sample that contain three or fewer nestlings.

Fill in the boxes below to define Y, indicating the type of distribution and its associated parameters.

$$Y \sim \square\left(\square, \square\right)$$

(2 marks)

(ii) Calculate the probability that more than half of the nests in the sample contain three or fewer nestlings.

(2 marks)

(iii) Find the mean and standard deviation of Y.

(2 marks)

(iv) Comment on what your results from parts (d)(ii) and (d)(iii) suggest about the likelihood that entire families of kookaburras in this region will survive a period of drought.

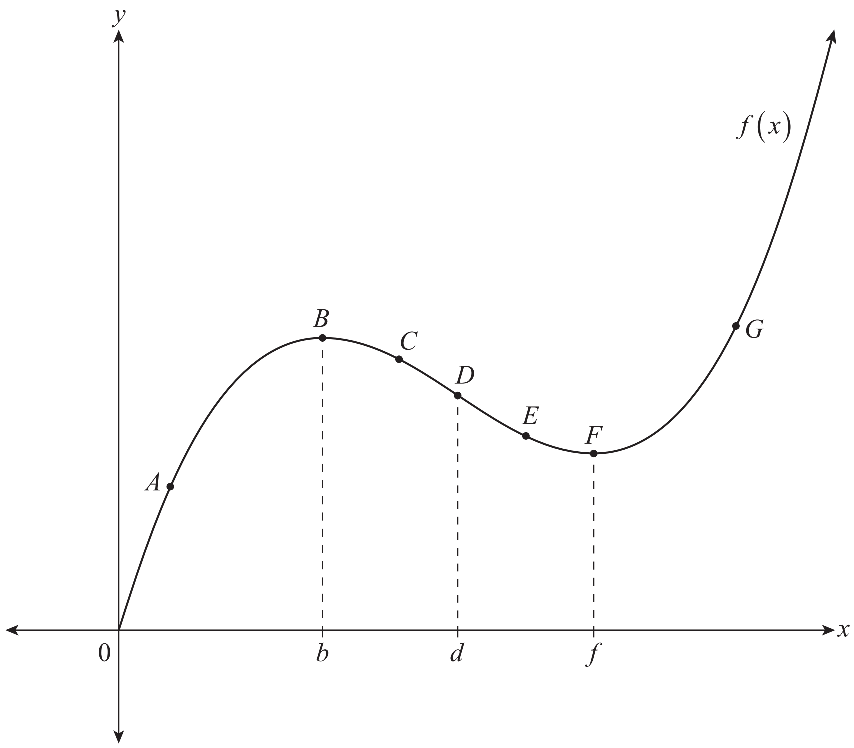
(2 marks)

Question 6 (5 marks)

Consider the function $f(x)$, defined for $x \geq 0$. The graph of $y = f(x)$ is shown below.

Point B is a local maximum with x-coordinate b, point D is an inflection point with x-coordinate d, and point F is a local minimum with x-coordinate f.

Points A, C, E and G are also shown on the curve.



(a) For what value(s) of x is $f(x)$ decreasing?

(1 mark)

(b) Refer to the seven labelled points on the graph on page 12.

Identify the point(s) with the following properties:

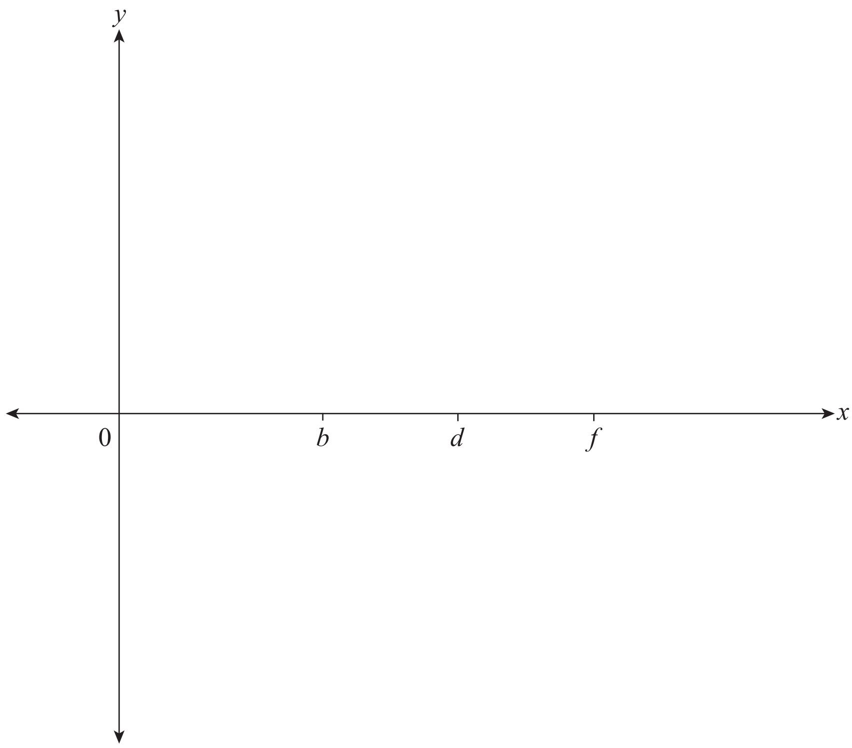
(i) $f'(x) > 0$ and $f''(x) < 0$.

(1 mark)

(ii) $f'(x) < 0$ and $f''(x) < 0$.

(1 mark)

(c) On the axes below, sketch the graph of $f'(x)$.



(2 marks)

PLEASE TURN OVER

Question 7 (5 marks)

Electricity usage is measured in kilowatt-hours (kWh).

In 2014 and 2015, the amount of electricity used by households in 1 week was studied. A standardised survey was used, and the number of kWh used in that week by each surveyed household was recorded.

In 2014, 1000 households were randomly selected and surveyed. The standard deviation of the weekly household electricity usage was 26.5 kWh.

In 2015, a further 50 households were randomly selected and surveyed. The mean weekly electricity usage for these 50 households was 126.3 kWh.

Assume that the standard deviation stayed the same between 2014 and 2015, and that household electricity usage is normally distributed.

(a) Using the information provided above, fill in the boxes below to form an expression that could be used to calculate a 95% confidence interval for the mean weekly household electricity usage in 2015.

$$\square - \square \times \square \leq \mu \leq \square + \square \times \square$$

(2 marks)

Assume that the 95% confidence interval for the population mean of weekly household electricity usage in 2015 is

$$118.95 \leq \mu \leq 133.65.$$

(b) Which *one* of the following statements is a correct interpretation of this 95% confidence interval? Circle the letter corresponding to the correct statement.

J The population mean is between 118.95 kWh and 133.65 kWh for 95% of weeks.

K If the survey is conducted 50 times, approximately 45 of the 50 resulting confidence intervals can be expected to contain the population mean.

L There is a 95% chance that the population mean is between 118.95 kWh and 133.65 kWh.

M None of J, K, or L.

N All of J, K, and L.

(1 mark)

In 2018, a further 100 households were randomly selected and surveyed to find the amount of electricity used in 1 week. The mean weekly electricity usage for these 100 households was 133.8 kWh.

Assume that the standard deviation stayed the same as during the 2014–15 period, and that household electricity usage is normally distributed.

(c) Based on the data, the media claimed that the population mean of weekly household electricity usage increased between 2015 and 2018.

(i) Calculate the 95% confidence interval of the 2018 survey data.

(1 mark)

(ii) Based on your calculation in part (c)(i), can the media's claim be supported? Justify your answer.

(1 mark)

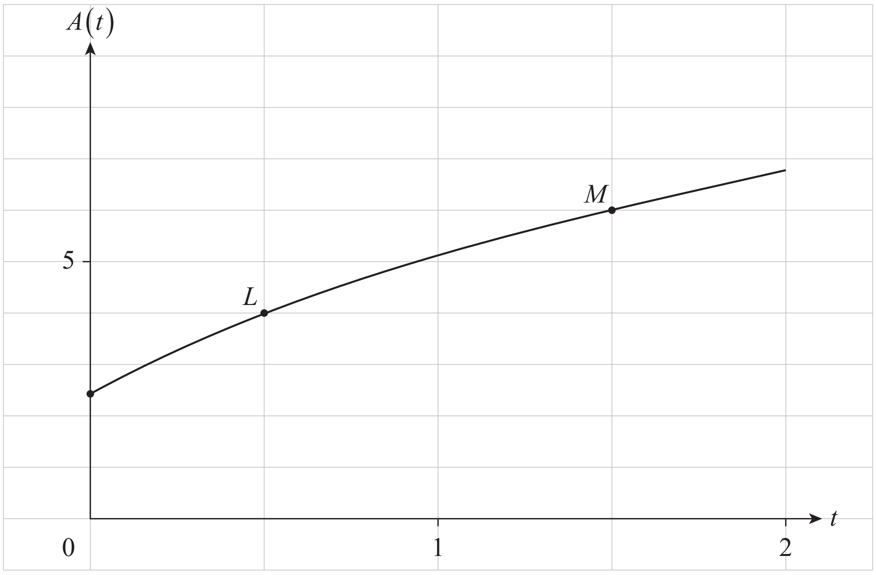
PLEASE TURN OVER

Question 8 (12 marks)

The area, in cm^2, of a colony of bacteria growing on a Petri dish after t hours can be modelled by the function:

$$A(t) = \sqrt{20t + 6}, \text{ for } t \geq 0.$$

The graph below displays the area of the colony over the first 2 hours. Points L and M are shown, representing $t = 0.5$ hours and $t = 1.5$ hours.



(a) Find the average rate of change of the area of the colony between L and M.

(2 marks)

(b) (i) Consider a further point, N, where the average rate of change of the area of the colony between M and N is a better approximation of the instantaneous rate of change at $t = 1.5$ than your answer to part (a).

On the graph above, plot and label a potential point N. (1 mark)

(ii) Calculate the average rate of change of the area of the colony between M and N.

(2 marks)

(c) Consider the general case at time t and at time $t + h$.

Write an expression for the average rate of change of the area of the colony between t and $t + h$.

(1 mark)

(d) Hence, using first principles, find the instantaneous rate of change of the area of the colony at any time.

(5 marks)

(e) State the instantaneous rate of change of the area of the colony at $t = 1.5$ hours.

(1 mark)

Question 9 (11 marks)

Logistic models can be used to understand and plan for change management in a workplace.

One workplace with 100 employees is planning to introduce a new software system called Lisen. Before Lisen is introduced, a small number of employees will undergo training to become skilled at using it. These employees will be 'initially skilled' at using Lisen, and will support others in the workplace to become skilled at using it.

The number of employees, N, who are expected to be skilled at using Lisen t weeks after it has been introduced can be modelled by the equation below.

$$N = \frac{100}{1+9e^{-0.4t}}$$

(a) State the number of employees who are expected to be initially skilled at using Lisen.

(1 mark)

(b) State the number of employees who are expected to be skilled at using Lisen 8 weeks after it has been introduced.

(1 mark)

(c) Find an expression for $\frac{dN}{dt}$.

(2 marks)

(d) Determine when the number of employees who are expected to be skilled at using Lisen is increasing at its greatest rate. Give your answer in weeks, correct to one decimal place.

(1 mark)

The managers of the workplace wish to schedule a training day, on which all employees will be trained in using Lisen.

For the training day, the managers would like the employees to be placed in groups of five, where each group contains at least one employee who is skilled at using Lisen.

The senior manager sets the following target:

'The probability that a randomly selected group of five employees contains at least one employee who is skilled at using Lisen should be at least 0.9'.

(e) A junior manager suggests that the training day can occur once 20% of employees are skilled at using Lisen.

Show that if the training day occurs when 20% of employees are skilled at using Lisen, the senior manager's target will ***not*** be met.

(2 marks)

(f) (i) Calculate the minimum proportion of employees who would need to be skilled at using Lisen in order for the senior manager's target to be met.

(3 marks)

(ii) Hence, state the minimum number of weeks after the introduction of Lisen that this training day would need to occur.

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 1. Make sure to label each answer carefully (e.g. 9(f)(i) continued).

South Australian Certificate of Education

Mathematical Methods

2019

Question booklet 2

- Questions 10 to 15 (74 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 18 if you need more space
- Allow approximately 95 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

SEQ	FIGURES	CHECK LETTER	BIN

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1. Brand ____________

Model ____________

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Question 10 (9 marks)

For positive integer values of n, the function $f(x) = kx^n(1-x)$ forms a probability density function on the interval $0 \le x \le 1$ for a certain integer value of k. For this value of k, $f(x) \ge 0$ for $0 \le x \le 1$.

(a) For $n = 1$, algebraically find the value of k such that $f(x)$ forms a probability density function for $0 \le x \le 1$.

(3 marks)

(b) (i) Find the area under the curve of $y = x^2(1-x)$, for $0 \le x \le 1$.

(1 mark)

(ii) Hence find the value of k such that $f(x) = kx^2(1-x)$ forms a probability density function for $0 \le x \le 1$.

(1 mark)

After considering several more values of n, the following conjecture is made:

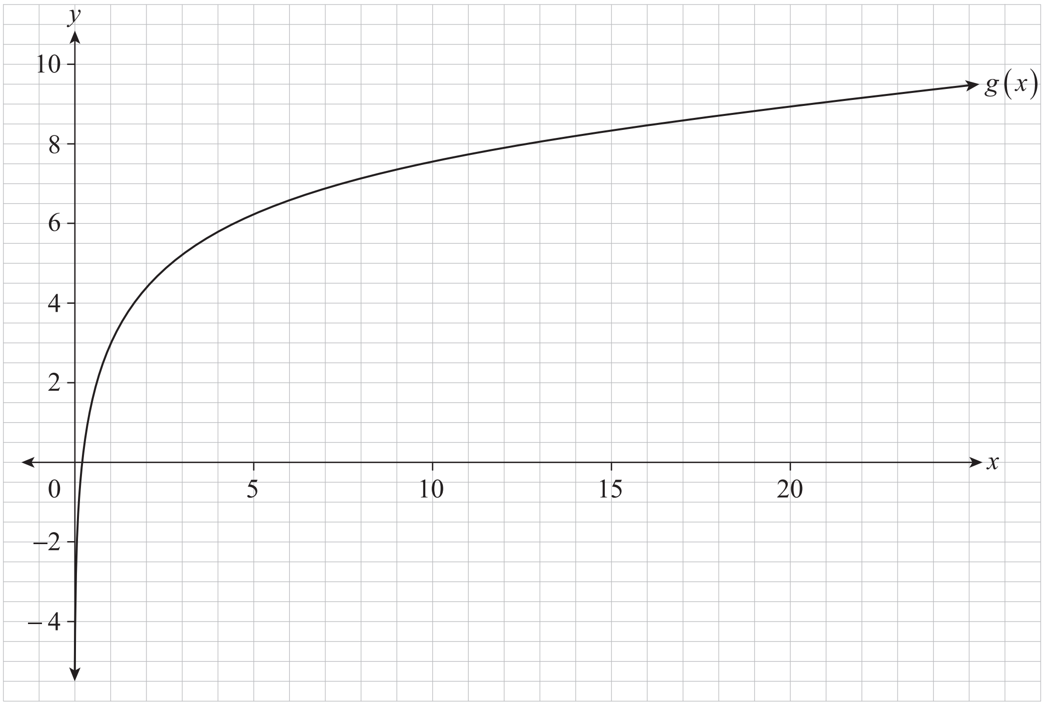
'In order for $f(x) = kx^n(1-x)$ to form a probability density function for $0 \le x \le 1$, $k = (n+1)(n+2)$'.

(c) Prove or disprove this conjecture.

(4 marks)

Question 11 (16 marks)

The graph of $y = g(x)$ is shown below, where $g(x) = 2\ln x + 3$ and $x > 0$.



Let $f(x) = (\ln x)^2$.

(a) On the axes above, sketch the curve of $y = f(x)$. Clearly show the coordinates of any intersection points or turning points. (3 marks)

(b) Using algebra, show that the solutions to the equation $f(x) = g(x)$ are $x = \frac{1}{e}$ and $x = e^3$.

(3 marks)

Let $D(x)=g(x)-f(x)$.

(c) (i) Calculate the value of $D(3)$.

(1 mark)

(ii) Interpret your answer to part (c)(i), in relation to the graphs of $y=f(x)$ and $y=g(x)$.

(1 mark)

(iii) State the domain of $D(x)$ if $D(x)\geq 0$.

(1 mark)

(d) (i) Show that $D'(x)=\dfrac{2-2\ln x}{x}$.

(1 mark)

Question 11 continues on page 6.

(ii) Determine $D''(x)$.

(2 marks)

(iii) Use $D'(x)$ to calculate the *exact* maximum value of $D(x)$.

(3 marks)

(iv) Use your answer to part (d)(ii) to justify that your answer to part (d)(iii) is the maximum value of $D(x)$.

(1 mark)

Question 12 (12 marks)

In 2016, the Vee Arr company began developing virtual reality goggles to use with computer games. However, some people experienced dizziness after using the goggles for an extended period of time.

In order to test how many people would be affected, Vee Arr decided to conduct a survey in which people used the goggles for 60 minutes and were asked if they experienced dizziness.

Vee Arr planned to sell its goggles if the proportion of people who experienced dizziness after using the goggles was less than 15%.

Of 100 people surveyed in 2016, a total of 26 people experienced dizziness.

Source: adapted from © Poike | iStock.com

(a) Calculate a 95% confidence interval for the population proportion of people who experienced dizziness after using the goggles for 60 minutes.

(2 marks)

(b) Vee Arr decided not to sell its goggles during 2016.

Explain why the confidence interval that you calculated in part (a) supports this decision.

(1 mark)

Question 12 continues on page 8.

(c) In 2019, Vee Arr developed a new model of goggles. They conducted another survey and found that of 110 people in the sample, 10% experienced dizziness after using these goggles for 60 minutes.

The 95% confidence interval of the data was:

$$0.0439 \leq p \leq 0.156.$$

(i) Determine the number of people from the sample of 110 who experienced dizziness when using the 2019 goggles.

(1 mark)

(ii) Vee Arr claimed that fewer people are likely to experience dizziness when using the 2019 goggles than when using the 2016 goggles.

Based on the confidence interval that you calculated in part (a) and the confidence interval given in part (c), justify whether or not this claim can be supported.

(1 mark)

Vee Arr still planned to sell its goggles if the proportion of people who experienced dizziness after using the goggles was less than 15%.

(iii) Does the confidence interval given in part (c) suggest that the 2019 goggles could be sold? Explain your answer.

(2 marks)

(iv) Assume that the sample proportion that experienced dizziness remained fixed at 10%.

(1) Explain why, if the width of the confidence interval given in part (c) was less than 0.1, Vee Arr could justify selling its goggles.

(2 marks)

(2) Hence or otherwise, find the minimum number of people who would need to be surveyed (10% of whom experience dizziness) in order to construct a 95% confidence interval with a width that is less than 0.1.

(3 marks)

Question 13 (11 marks)

Frankie is an electrician based in Goodtown. She is considering introducing a new business model in which customers would pay a fixed price of 250 dollars ($) to purchase a ceiling fan and have it installed. Frankie knows that the actual cost of purchasing and installing a ceiling fan varies according to the type of material that the house is made from, as detailed in the table below.

Type of material	*Wood*	*Brick*	*Stone*	*Other*
Cost to electrician ($)	150	190	250	280

To investigate the likelihood of a long-term profit for this business model, Frankie carries out some research and finds that in Goodtown, 10% of houses are made from wood, 40% of houses are made from brick, and 20% of houses are made from stone.

Frankie assumes that Goodtown-based customers who contact her will live in 'randomly selected' types of house. Hence, let the amount of profit (in $\$$) received by Frankie per house be represented by the discrete random variable X.

(a) Based on the information provided, complete the discrete probability distribution table below for X.

Type of material	*Wood*	*Brick*	*Stone*	*Other*
x	100	60		
$\Pr(X = x)$	0.1	0.4		

(2 marks)

(b) (i) Hence calculate the value of $E(X)$.

(1 mark)

(ii) Interpret your answer to part (b)(i) in the context of the long-term profit.

(1 mark)

(c) Calculate σ_X.

(1 mark)

(d) Frankie can install a maximum of 15 ceiling fans per week during the 48 working weeks in 1 year.

(i) What is Frankie's maximum expected yearly profit from installing ceiling fans?

(1 mark)

(ii) Give *one* reason why Frankie's maximum expected yearly profit might statistically differ from her actual yearly profit.

(1 mark)

(e) Frankie requires a profit of at least $\$50\,000$ per year in order for this business model to be worthwhile.

(i) What fixed price would Frankie need to charge, in order for the business model to be worthwhile?

(3 marks)

(ii) Suggest *one* change that Frankie could make to the business model in order to increase her expected yearly profit (other than changing the fixed price).

(1 mark)

Question 14 (12 marks)

Consider the curve defined by $y = f_k(x)$ where

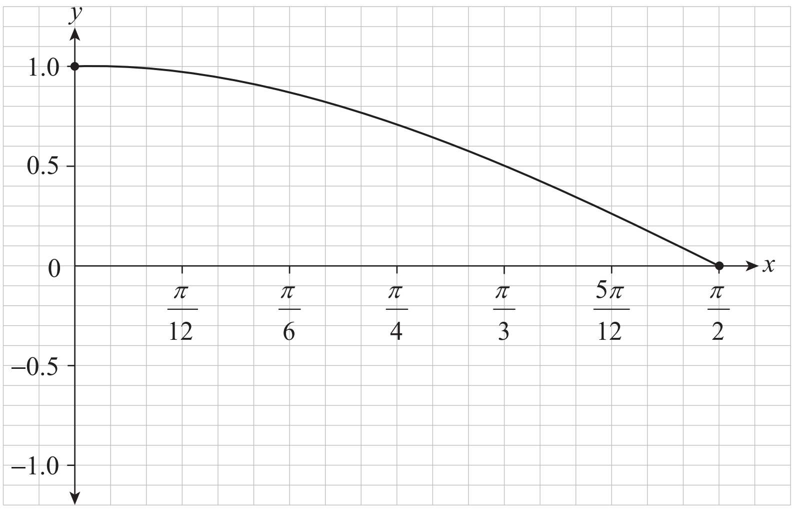
$$f_k(x) = \cos kx$$

and k is a positive integer.

The domain of the function $f_k(x)$ is dependent on the value of k, such that $0 \le x \le \frac{\pi}{2k}$.

(a) If $k = 1$ for $y = f_k(x)$ then the function is $f_1(x) = \cos 1x$.

The corresponding domain for the function of $f_1(x)$ is therefore $0 \le x \le \frac{\pi}{2}$. The graph of $f_1(x)$ is shown below.



Using algebra, calculate the area between $y = f_1(x)$ and the x-axis over the domain of the function.

(3 marks)

(b) Let $k = 2$ for $y = f_k(x)$.

(i) State $f_2(x)$ as a function.

(1 mark)

(ii) State the domain of the function $f_2(x)$.

(1 mark)

(iii) On the axes on page 12, sketch the curve of $y = f_2(x)$ over the domain of the function $f_2(x)$.

(1 mark)

(iv) Calculate the area between $y = f_2(x)$ and the x-axis over the domain of the function.

(1 mark)

(c) Find the value of $\int_0^{\frac{\pi}{6}} \cos 3x \, dx$.

(1 mark)

(d) (i) Make a conjecture about the exact value of the area bounded by $y = f_k\left(x\right)$ and the x-axis over $0 \leq x \leq \frac{\pi}{2k}$, for any value of k.

(1 mark)

(ii) Prove or disprove your conjecture.

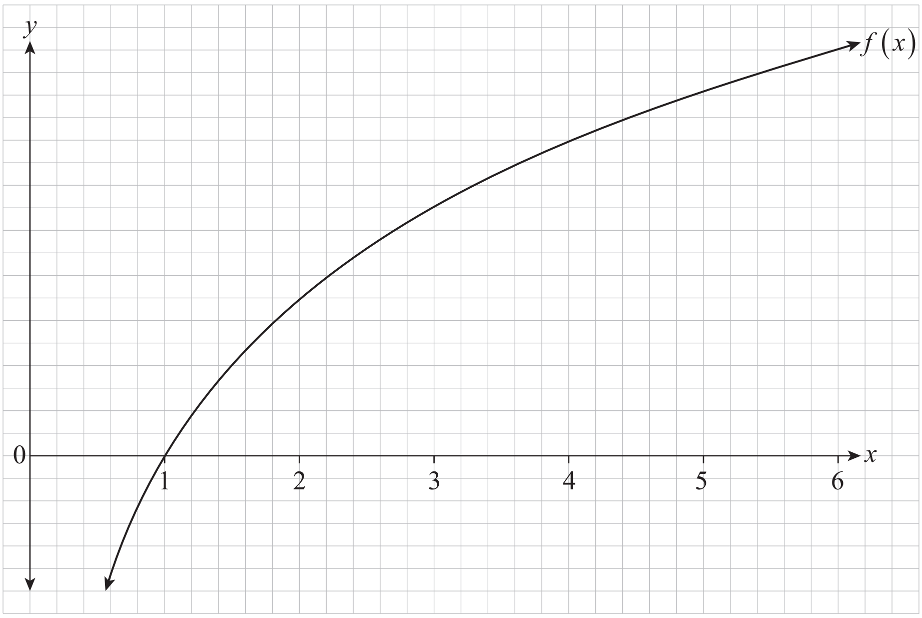
(3 marks)

Question 15 (14 marks)

(a) Show that, if $y = x\ln x - x$, then $\dfrac{dy}{dx} = \ln x$.

(2 marks)

Consider the function $f(x) = \ln x$. The graph of $y = f(x)$ is shown below for $x > 0$.



(b) An overestimate of the area between the graph of $y = f(x)$ and the x-axis from $x = 1$ to $x = 5$ is to be calculated, using four rectangles of equal width.

(i) On the graph above, draw the four rectangles used to determine this overestimate.

(1 mark)

(ii) Calculate this overestimate, giving your answer as an *exact* value.

(2 marks)

(c) Consider the definition that $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.

Explain why $\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$
$= \ln 2 + \ln 3 + \cdots + \ln n.$

(1 mark)

(d) (i) Find an expression for an overestimate of the area between the graph of $y = f(x)$ and the x-axis from $x = 1$ to $x = n$, using $n-1$ rectangles of equal width.

(1 mark)

(ii) Hence, explain why $\ln n! > \int_1^n \ln x \, dx$.

(2 marks)

(iii) Using the information in part (a), evaluate $\int_1^n \ln x \, dx$.

(2 marks)

(iv) Hence, use the inequality given in part (d)(ii) and your answer to part (d)(iii) to show that

$$n! > n^n \times e^{1-n}.$$

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 12(e)(i) continued).

MATHEMATICAL METHODS FORMULA SHEET

Properties of derivatives

$$\frac{d}{dx}\left(f(x)\,g(x)\right)=f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

$$\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$$

Quadratic equations

If $ax^2+bx+c=0$ then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X=\sum xp(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X=\sqrt{\sum\left[x-\mu_X\right]^2 p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X=k)=C_k^n p^k(1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p}=\frac{X}{n}$,

where a sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\le p\le\hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X=\int_{-\infty}^{\infty}x f(x)\,dx,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X=\sqrt{\int_{-\infty}^{\infty}\left[x-\mu_X\right]^2 f(x)\,dx},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with mean μ and standard deviation σ is:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu=0$ and $\sigma=1$ by:

$$Z=\frac{X-\mu}{\sigma}.$$

Sampling and confidence intervals

If $\bar{x}$ is the sample mean and s the standard deviation of a suitably large sample, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x}-z\frac{s}{\sqrt{n}}\le\mu\le\bar{x}+z\frac{s}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.